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ROLE OF MATHEMATICS IN QUANTUM THEORY

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by

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1. Introduction

Mathematics has always played a significant role in many branches of Physics, but, it is correct to say that this role has been predominant in quantum theory, specially when taken in conjunction with relativity theory. In fact, the most abstract notions in mathematics developed on a purely axiomatic basis using ingenious logical constructions, with a view to derive beautiful results of great generality and simplicity, appear to afford an explanation of many of the complex phenomena studied in quantum mechanics. Moreover, it is often found that some of these explanations are of such unexpected and fantastic accuracy as to suggest the intervention of a supernatural agency. Equally remarkable are the facts that this is not true of all mathematical concepts, and that even such miraculously successful theories are only of limited scope, and can at best be considered as highly useful tools.

Mathematics is a type of formal logic based on axioms and rules for the construction of operations applied to the entities postulated in the axioms, and finally leads to theorems belonging to various levels of complexity. The choice of the axioms and the rules of operation is governed by an aesthetic sense, and often results in theorems constituting supremely beautiful free mathematical creations. Not all such creations have, however, found applications in physics, nor has it been possible to foresee which of these mathematical techniques are likely to be successful in the correct formulation of the laws of physics. Sometimes, it has even happened that many mathematical theories built by the theoretical physicist solely keeping this success in view have been fruitless, consequent on the fact that they have been found inadequate to explain a new law nature suddenly and unexpectedly discovered. A recent example of this is the striking parity and β-decay revolution of 1957-58 which has been a little embarrassing to the theorist by showing that mere abstractness and

mathematical elaboration for their own sake do not always yield fruitful results, and that a phenomenological theory may be extremely valuable at times. This does not, of course, in anyway minimise the importance of the role of mathematics in quantum theory, but only emphasises the mysterious nature of this role. I wish to indicate below a few of the striking roles of revolutionary significance played by mathematics in quantum theory.

2. Mathematical foundations of quantum theory

One of the most beautiful examples showing the importance of mathematical concepts in formulating the laws of quantum mechanics is given by the building up of an axiomatic basis for these laws, as done explicitly by J. Von Neumann, and implicitly by Dirac. The essence of this basis is the setting up of the two fundamental concepts of states and observables in quantum mechanics in the language of the infinite-dimensional complex Hilbert space. The states are vectors in this space, the observables are self-adjoint operators on these vectors, and the possible values of the observables are the characteristic values of the operators. The initial steps in the Hilbert space formulation of Von Neumann consist in presenting the Dirac-Jordan, and Schrodinger developments of the transformation theory unifying the matrix and wave aspects of quantum mechanics, putting the former development on a proper axiomatic basis by associating a discrete function space F_z with it, associating a continuous space F_Ω with the latter development, next showing that F_z and F_Ω are isomorphic, and finally building up an unique abstract Hilbert space H on an axiomatic basis, with properties belonging to F_z as well as to F_Ω. The further development of the geometry of H based on the five axioms relating to linearity, definition of an inner product which is Hermitian, dimension, completeness, and separability resulting in nine fundamental theorems, the introduction of closed linear manifolds and several types of operators in this space, specially the Hermitian ones, a study of the central eigen-value problem with emphasis on the existence and uniqueness of its solution, as well as its nature in the case of different kinds of operators, and lastly the establishment of a rigorous foundation of the concept of the trace of an operator constitutes a mathematical formalism of unparalleled beauty and logical precision. Following up this geometrical formulation, Von Neumann develops the physical aspects of quantum mechanics by deducing rigorously the fundamental basis of the

statistical theory, and clarifying the notions of statistical interpretation, measurability, projections, and the uncertainty relations. As an illustration of the axiomatic method based on the Hilbert space approach, he has also given a derivation of the radiation theory advanced by Dirac, which is one of the most beautiful achievements in the field of quantum mechanics.

Further developing the notion of projections in H as experimental propositions constituting subsets of H considered as a phase-space, Birkhoff and Neumann have set up a logic applying to these propositions, and shown that it is possible to build up a propositional calculus for quantum mechanics based on the facts that all operators are Hermitian, and that all Hermitian symmetric operators in H correspond to observables. Proceeding on this basis, they have derived the interesting result that this propositional calculus has the same structure as an abstract projective geometry. By way of contrast, they have examined the nature of experimental propositions in classical statistical mechanics, and assuming that these correspond to Lebesgue measurable subsets of the corresponding phase-space, they have shown that this propositional calculus constitutes a Boolean algebra.

It is also worth noticing that the essentials of Von Neumann's geometric treatment have survived the two great extensions which quantum theory has undergone *viz.*, the relativistic quantum mechanics, and the generalised quantum theory of fields, at least for the theory of non-interacting fields.

Mention may also be made here of the parallel work of M. H. Stone on linear transformations in H, specially on the algebra of such transformations. He has pointed out that Klein's characterisation of geometry in terms of the theory of groups finds application in the geometry of H, and shown, in particular, that the class of all bounded linear-transformations with domain H, which possess bounded inverses also with domain H, and the class of all unitary transformations in H, both constitute groups. He has further set up the isomorphism of the former group with the ordinary matrix algebra with infinite matrices, and made valuable contributions towards solving the problem of unitary equivalence of self-adjoint transformations.

That the Hilbert-space approach is not a mere mathematical formalism incapable of being employed with suitable modifications for the interpretation of the newer developments of quantum mechanics is shown by some very recent work of J. Schwinger who has considered afresh the geometry of quantum states, and shown that the

algebra of measurement symbols can be derived from the geometry of an N-dimensional space of states or of an N^2-dimensional space of operators. It would be interesting to examine the connection of these spaces with Von Neumann's abstract H satisfying his axioms A, B, D, E, and $C^{(n)}$ instead of $C^{(\infty)}$.

3. Field Quantisation

The radiation theory mentioned above as developed by Von Neumann, or what is equivalent to it, the Jordan-Pauli method of quantisation for free boson fields, and the Jordan-Wigner method of quantisation for free fermion fields are both well-known techniques giving satisfactory detailed descriptions of the wave-particle dualism for both types of fields. But these methods, however, are not Lorentz covariant, and are not particularly straight forward and elegant since they are based on a formal extension of the Poison-bracket method. Powerful methods of field quantisation without these drawbacks, and using more abstract mathematical ideas, have been set up by Schwinger and Feynman, which we will briefly present here. These methods have the additional advantage of being applicable to the case of interacting fields also.

Both these methods are based on the principle of least action of classical mechanics, which states that for an actual trajectory between two points, the principal function W is a function of the co-ordinates, momenta and time only of the end points, and the well-known result that W defines a canonical transformation between the co-ordinates and momenta corresponding to the end points. Using the relation obtained by Dirac between canonical transformations in classical and quantum mechanics, it becomes possible to set up an operator which plays the role of an action function W in quantum mechanics also. The essential problem is to construct this operator suitably so as to make the corresponding transformation-function unitary in the Heisenberg representation. This unitarity results if the action operator can be constructed so as to be Hermitian, and remarkably enough, this cannot be achieved if one tries to construct a finite W but becomes possible by working with an infinitesimal δW as shown by Schwinger, or by replacing the finite transformation function by a decomposition into products of transformations each with an infinitesimal variation in the end times appearing therein, as shown by Schwinger.

The first step in Schwinger's method thus consists of setting up a δW, in the relativistic form, of the variation of an integral whose

integrand is a Lorentz invariant Lagrangian density which is a Hermitian function of the Hermitian field amplitudes and their first derivatives, and at most bilinear in these derivatives. The next step is the writing down of a fundamental dynamical principle for quantised fields, analogous to the classical action principle, which enables the expression of the variation of the corresponding transformation in terms of δW. Schwinger has shown that both these steps can be satisfactorily realised, and that this method of field quantisation can be applied successfully both to boson and fermion fields, but that there still remain difficulties of interpreting some of the mathematical results obtained.

Feynman's principle, which, instead of taking the operator as fundamental takes the transformation function as fundamental, uses the representation of this as the product of a large member n of such functions, requires mathematical ideas more abstract than those used in Schwinger's method. Without going into the actual details of Feynman's method, it might be remarked that the classical analogue of this method is provided by a remark of Dirac that the action principle of classical mechanics is equivalent to the statement that the action is stationary for variations of the intermediate values of the generalised co-ordinates about the actual classical trajectory. Further, in Feynman's method the formal expression for the product transformation function in terms of the action operator consists of a multiple integral in a non-enumerable number of dimensions which of course has no real physical significance. Feynman has got over this difficulty by choosing for the time-dependent functions which give the set of paths constituting the actual trajectories in the action principle a set of independent normal orthogonal differentiable functions of t obeying suitable conditions so that the paths lie in a Hilbert space defined by the set of orthogonal functions. After this modification the final expression for the transformation function is obtained by going to the limit as $n \to \infty$, and leads to a *functional integral* i.e., one in which the variables with respect to which an integration is performed are replaced by functions of these variables. There appear further difficulties in applying Feynman's method to boson and fermion fields and the difficulties of physical interpretation in the latter case have not yet been resolved.

I have mentioned the above two methods of field quantisation in order to emphasise a special aspect of the role of mathematics in quantum theory, *viz.* that any quantum-mechanical problem, whatever its complexity can be associated with a rigorous mathematical

formalism, which, however, may not serve the purpose of throwing light on the problem, consequent on the difficulty of interpreting the mathematical results obtained.

4. S-Matrix and Renormalisation

Although the two general methods of field quantisation mentioned above can, in principle, be applied to interacting fields also, there are very few cases where they have actually been so used successfully. The case of interacting fields gives rise to peculiar difficulties of a mathematical type. In the case of free fields, the field operators can be set up in a separate Hilbert space in which the state vectors can be normalised without any difficulty. There are also a few problems relating to interacting fields which can be solved exactly by using similar methods. But, in general, difficulties have appeared in attempts made to give a rigorous mathematical formulation for interaction fields, and raised the question whether the customary use of operators, state vectors and unitary transformations in Hilbert space can be justified in all such cases. In fact, there are examples where any state vector of the non-interacting system is orthogonal to all state vectors of the interacting system, thus making the H-space picture meaningless. In other words, it becomes impossible in such cases to apply the usual perturbation theory for interacting fields, since the validity of this theory depends on the interaction between the two systems producing only a small change in the state vectors of the systems. Otherwise, the perturbation theory may give infinite results, and hence it becomes necessary to examine, in every case of interacting fields, whether the perturbation theory can be applied so as not to lead to such divergences. The cases where such divergences can be avoided constitute the so-called re-normalisable theories, and the modification of the perturbation method towards this end is called *renormalisation.* The possibility or otherwise of renormalisation depends on the nature of the interaction Hamiltonian H_{int} which we might, in general, assume to vary with time, along with the two free Hamiltonian terms, H_1 and H_2 also. The possible cases arise when we can find a suitable matrix which describes transitions from one set of eigenstates of $H_0\,(=H_1+H_2)$ to another set of eigenstates of H_0 consequent on the interaction. Such a matrix or depending on the interaction operator can be shown to exist when bound states of the interacting system do not occur, and can be written in the form $U(t, t_0)$ where t_0 is an arbitrary fixed time and both t and t_0 are finite, and moreover $U(t, t_0)$ is a unitary operator. If bound states

occur, the corresponding matrix to be considered is $U(\infty, -\infty)$ which can also be shown to be unitary when it exists. $U(\infty, -\infty)$ is called the *S-matrix*, so that a renormalisable theory is possible when a unitary S-matrix can be set up.

An example of a renormalisable theory is afforded by the usual quantum electrodynamics. Here the electron in a hydrogen atom interacts with the Coulomb field, and in addition, it interacts with the quantised electomagnetic field by emitting and absorbing photons. A straightforward perturbation theory calculation would give infinite results but it can be shown that an S-matrix can be found permitting renormalisation in this case, which amounts to giving a finite value, agreeing with experiments, to the change in the electron's energy due to the virtual emission and absorption of photons as the difference between two divergent integrals. From the mathematical point of view, however, renormalised quantum electrodynamics does not appear to be quite satisfactory since it does not set up a meaningful Hilbert space picture by giving a self-consistent scheme of operators and normalised state-vectors describing the interacting quantised electron and Maxwell fields. It is perhaps not satisfactory to the physicist also since it appears like a mathematical trick subtracting infinity from infinity and getting a finite result. The methods of renormalisation which have been applied to circumvent the appearance of infinities in practical applications have also not solved the fundamental question whether such infinities were due to the inadequacy of the mathematical methods or were rather intrinsic in the physics itself. Some attempts recently made to show that the former alternative is the correct one, have used algebraic theorems on the expansions of pfaffians, and permanents (*i.e.*, determinants expanded with a positive sign rule), combinatorial models, Hadamard's concept of *parte finie* of divergent integrals, and Hilbert's integral equation, but it cannot be said definitely that the mathematical difficulties have been successfully overcome. Many other difficulties of both a mathematical and physical nature have been pointed out by various workers in the field of renormalisable theories, and the Lorentz-invariant form of it indicated above appears to succeed only for weakly interacting fields, like the electrodynamic case.

In spite of the unsatisfactory features mentioned above, the renormalised quantum electrodynamics, and other similar theories provide a detailed physical interpretation of all the various individual physical phenomena contained in the interaction, by a suitable use of the associated S-matrix. For example, this matrix for electrodynamics can be arranged in the form of an infinite series, the terms

of which are successively simple, double, and multiple infinite integrals of range $-\infty$ to $+\infty$. The integrands of these can be arranged as products in a physically meaningful order by the use of a chronological operator F introduced by Dyson, and a normal ordering operator N for the factors of a product. Then each of the terms or elements of the S-matrix can be sorted out using this ordering theorem, and this sorting out illustrates the various physical phenomena contained in the term. Thus, using the second or double integral term S_2, the photon component gives two terms, and the electron-positron component gives four terms, and multiplication of these gives eight terms in all representing the interaction processes. Further, these processes can be illustrated graphically by using suitable conventions regarding solid and broken lines for fermion and photon contractions respectively, and one thus obtains six *topologically distinct* graphs describing vacuum state fluctuations, and the real processes like the Moller scattering, Compton effect (and pair annihilation and creation), fermion self-energy, and photon self-energy. An elaboration of this graphical description can be made by making the further convention of regarding positrons as electrons moving backward in time, which leads to no ambiguity in interpreting the graphs, and enables one to distinguish electron and positron processes. Such an elaboration gives rise to what are called Feynman graphs, each of the graphs mentioned earlier giving rise to several other Feynman graphs. Thus, for example, the Moller scattering graph gives rise to four Feynman graphs, each of them again topologically distinct from the other three, showing electron-electron, positron-positron, and electron-positron scatterings. This appearance of topological notions in interpreting graphs representing physical processes clarifies the study of these interactions to a great extent. Even higher order processes using terms like S_4 in the S-matrix series can be represented by topologically distinct graphs which would correspond, when S_4 is used, to notions of electron and positron self-energies, and scattering of light by light. Much work has recently been done in analysing several of these graphs so as to interpret geometrically the notion of renormalisation itself. Towards this end, attempts have also been made to obtain compact formulae which will collect together, for any perturbation order, large numbers of topologically equivalent Feynman graphs so as tosimplify the problems of enumeration and classification.

5. Dispersion Relations

Complex numbers, and the theory of functions of a complex variable have earlier played indirect roles in classical physics as shown, for example, by the use of conformal transformation techniques in electro-magnetic theory. The use of complex numbers in quantum mechanics, however, appears more direct as shown by the fact that their use comes close to being a necessity in the formulation of the laws of quantum mechanics based on the complex Hilbert space with a Hermitian scalar product. A suggestion, based on a physical basis has recently been made that the assumption of the vector-space of quantum mechanics being a complex space is necessary to explain the fact that, in quantum theory, a double infinity of states results by superposition from two given single states, *i.e.*, a particular superposition is specified only by giving *two* real parameters, as for example, relative weight and phase of the state. That this is the actual state of affairs is shown by the interference of de Broglie waves, by elliptic polarisation state of photons, by the completeness of the set of eigen-states for the eigen-values $+\frac{1}{2}, -\frac{1}{2}$ of a spin component of a Pauli electron, and also by the essential role which phase factors play in the quantum theory of measurement. An extension of complex field quantum mechanics has recently been attempted by setting up a quaternion quantum mechanics having the attractive features of affording an easier introduction of isotopic spin, and an explanation of the unique value of the elementary quantum of charge. Thus, it appears that a complex space or a generalisation of it is needed to explain charge properties of elementary particles.

More than the use of complex numbers, and complex spaces, recent work appears to show that analytic functions of a complex variable are destined to play a decisive direct role in the formulation of quantum theory. These functions are being extensively used in the rapidly developing theory of dispersion relations, specially in the field of elementary particle physics. These relations are integral formulae connecting the real and imaginary parts of the scattering amplitudes. Using the analogy of classical theory, we know that in optics, a dispersion relation is an integral relation connecting the real and imaginary parts of the refractive index $n(\omega)$. This is easily done by using a simple model in the classical case, examining the poles of the function $f(\omega)$ of the complex variable (ω), which gives the forward scattering amplitude at frequency ω, and expressing the real and imaginary parts of $f(\omega)$ as principal value integrals *i.e.*, as Hilbert transforms, constituting the dispersion relations. An alternative and

a more general way of obtaining these relations is to use the so-called causality relation that for an effect to take place, there must be a cause prior to it, such that the cause at the space-time point $(x, t) = 0$ cannot produce an effect at $x > 0$, $t > 0$ if $x > ct$, which condition means that no signal can travel faster than light. Putting this analytically, if the output signal $G(t)$ be expressed as a Fourier transform of the input signal $A(t)$, with $F(t)$ as the kernel, casuality requires $F(t) = 0$ for $t < 0$ and the Fourier transforms $g(\omega)$, $a(\omega)$ and $f(\omega)$ would be connected by $g(\omega) = f(\omega)\, a(\omega)$. Further if the integral of $|f(\omega)|^2$ converges, a well-known result due to Titchmarsh in the theory of Fourier integrals immediately leads to the dispersion relations.

It can be shown that these dispersion relations are suited for use in problems of scattering in quantum mechanics also, if we add the causality condition on the S-matrix in addition to the conditions of Lorentz invariance and unitarity mentioned in the previous article. The detailed analysis of the causality condition shows the need for imposition of further restrictions on the elements of the S-matrix, and these conditions imposed lead to the dispersion relations by applying the Cauchy integral theorem, or what is equivalent to it, the theorem of Titchmarsh mentioned above.

Mention may be made of dispersion relations obtained by using the Schrōdinger wave equation technique, by introducing the notion of complex orbital momenta. The orbital momentum j until now considered as an integral discrete parameter in the radial Schrōdinger wave equation, can be allowed to take complex values. This opens new possibilities of discussing the connection between potentials and scattering amplitudes. For special classes of potentials $V(x)$ which are analytically continuable into a function $V(z)$ $(z = x + iy)$, regular and suitably bounded in $x > 0$, the scattering amplitude has the remarkable property of being continuable for arbitrary negative, and large cosine of the scattering angle, and therefore for arbitrarily large and positive transmitted momentum. The range of validity of the dispersion relations is much enlarged, as can be shown by using a generalisation of a theorem due to Poincare on integral functions.

Dispersion relations have been obtained in quantum electrodynamics without the use of perturbation theory, but by using the general Heisenberg picture, and in a few other interactions also. In particular, the derivation of dispersion relations for pion nucleon scattering presents some interesting mathematical features, consequent on the existence of an unphysical region, $0 < w < \mu$ (μ = renormalised mass of the meson) which has to be excluded in the

evaluation of the integrals expressing the dispersion relations. The difficulty is the interpretation of this unphysical region in terms of physical data. Its meaning may not necessarily be expressible in terms of the physical S-matrix elements ; in fact, it has been shown that the notion of *analytic continuation* of such a matrix element clarifies the interpretation. Some recent general work examines dispersion theory on the basis of a reciprocal relationship between the real and imaginary part of the scattering amplitude, and shows that the knowledge of the absorptive part over the unphysical region leads to a linear *integral equation* for the dispersive part.

6. Algebra and Group Theory—Non-relativistic Quantum Mechanics

The theoretical techniques described in the last four sections constitute one particular kind leaning heavily on mathematical analysis. Other techniques like the numerous invariance and symmetry properties, and the selection rules constitute another kind using the methods of algebra and group theory. We shall discuss some of these properties and rules in this, and the next two sections.

The relationship between algebra and physics has been rather a loose one in the earlier years, and although group theory has played an important role in theoretical physics, it has mostly been a subordinate one. But in recent years, the more profound branches of group theory, and several branches of modern algebra have been freely used in quantum theory to elucidate fundamental questions. We have already mentioned the algebra of transformations in Hilbert space. Other examples of algebras may be mentioned. One such is the Jordan quantum mechanical algebra suggested for investigating statistical properties, which is commutative, but not associative unlike the ordinary non-commutative algebra of matrices of q numbers. The study of the structure of such Jordan or r-number algebras shows that all irreducible r-number algebras are equivalent (with a single exception) to the algebras obtained by quasi-multiplication of real matrices, where quasi-multiplication is defined by $ab = \frac{1}{2}(a.b + b.a)$ with $a.b$ denoting the ordinary matrix product. The exception is the algebra of all three-rowed Hermitian matrices with elements in the real non-associative algebra of Cayley numbers. The notions of rings and fields used in modern algebra can also be shown to have counterparts in quantum theory also. A simple example is that of operators in the abstract Hilbert space H of § 1 constituting a ring, and another is that of two-rowed complex-matrices constituting a field.

Coming to groups specifically, types of these of physical interest, specially in non-relativistic quantum theory, are the symmetric permutation groups, the quaternion group, the three-dimensional rotational group, and the two dimensional unimodular unitary group. These four types of groups have two common properties, *viz.*, that every element is equal to its reciprocal, and the Kronecker product of any two irreducible representations of the group contains no representation more than once. Groups defined by these properties are called simply *reducible*, and it can be shown that the groups related to most eigen-value problems occurring in non-relativistic quantum theory are simply reducible. The group concept arises in this theory from the fact that all the symmetry transformations of a Schrödinger Hamiltonian form a group. If the Hamiltonian H be invariant under a group G of transformations, then the eigen functions belonging to one energy level F form the basis of a representation D of G. If G includes all symmetry transformations of H, then D is irreducible apart from accidental degeneracy, and the application of a perturbation can also be put in group-theoretic form.

Considering special types of simply reducible groups, we have the results that all the representations of Abelian groups are one-dimensional, that permutation groups of any order have always the symmetric and anti-symmetric representations, and that the irreducible representations of the full rotation group can all be represented by $D_{(j)}$ (where j is an integer or half-odd integer) of dimension $2j+1$, with a wave function transforming according to $D^{(j)}$ describing a state of angular momentum $\sqrt{j(j+1)}$ in units of H. In view of its importance, we can deal with the last type of group in greater detail. It can be shown that in this case, the product representation $D^{(j)} \times D^{(j')}$ can be suitably reduced, and that this reduction finds application in the quantum mechanics of the free atom having the Hamiltonian $H = H_{orb} + H_{spin} + H_{nucl}$. The application to H_{orb} shows that it is invariant under simultaneous rotation of all electron co-ordinates, and also shows how the eigen functions transform under this. The application to the second part of H is really equivalent to introducing the notion of spin quantum mechanically as an internal degree of freedom of an electron described by a co-ordinate σ_z taking on only two values $\sigma_z = \pm 1$ (or the intrinsic angular momentum being $\pm \frac{1}{2}$), and the associated spin wave functions $U+$, $U-$ transforming into one another according to $D^{1/2}$. Considering $H_{orb} + H_{spin}$ under space inversions P only leads to the notion of *parity* which is + or – *i.e.*, even or odd, according as the eigen functions change to + or – of their

original values. Since the spin functions $U+$ and $U-$ have even parity and H_{orb} is invariant under P, it follows that $H_{orb} + H_{spin}$ is also invariant under P, so that the parity is also an exact quantum number. The invariance of this sum Hamiltonian under the group of permutations of electron and spin co-ordinates leads to the famous Pauli *exclusion principle* that the physically applicable wave functions are all antisymmetric in this case.

Going deeper into the theory of finite groups, we can introduce the notion of group characters to label representations, particularly irreducible representations, and also obtain types of orthogonality relations used to derive the characters of all these representations. Product groups can be introduced, and methods found to construct all their irreducible representations out of those of the componen groups. These considerations relating to finite groups enable one to sort out the eigen functions of H according to the irreducible representations of the symmetry group of H. Besides the space inversion P and the permutation transformations mentioned above, we can also discuss the transformation of time-reversal as a symmetry transformation of H for any electronic system, and show that it leads to additional degeneracies, in particular, giving Kramer's theorem that for an odd number of electrons all states have an even degeneracy. Finally, the consideration of H_{spin} in greater detail leads to results relating to nuclear spin and parity, which limit the number of possible non-zero electrical and magnetic multiple moments.

Coming from the electrons in an atom to nucleii, which contain two separate types of particles, protons and neutrons, the wave functions describing them become one stage more complicated, and leads to the notion of *isotopic spin*, a formalism which considers a proton and neutron as two states of one particle, the nucleon, distinguished by the isotopic spin quantum number $\tau_z = \pm\frac{1}{2}$, introduced by using a fictitious three-dimensional isotopic spin space. This leads to the use of a total isotopic spin vector, $\vec{T}$, the irreducible representations $D^{(T)}$ of the rotation group in this space, and the operator M_T which is a measure of the nuclear charge, with T and M_T as total quantum numbers for any system. In this formalism, the exclusion principle takes the form that a wave function must be antisymmetric under any permutation of the nucleon co-ordinates, the orbital, the spin and the isotopic spin co-ordinates also being simultaneously permuted. The isotopic spin formalism is specially suited to deal with the "strong interactions" or the specifically nuclear forces between nucleons, as contrasted with the electromagnetic interactions, and the "weak interactions" which couple nucleons

with electrons and neutrinos, and which are responsible for β-decay. These strong interactions have also the symmetry property of being invariant under P so that a parity number can be associated with each nuclear level. They are also charge independent which makes the Hamiltonian describing the strong interactions invariant under rotations in isotopic spin space. The second invariance enables the eigen functions of the Hamiltonian being sorted out to transform according to the irreducible representations $D_{(T)}$, and thus explains the existence of isotopic spin multiplets. Moreover, the same technique can be applied to other particles, as for example, the pion with charge states π^+, π^0 and π^- *i.e.*, $M_T = 1, 0, -1$, and $T = 1$ under isotopic spin rotations, and here also the interactions are of the "strong" type. In addition to T and M_T, such particles can be described in terms of two other quantum numbers, namely the *nucleon number N*, and another called the *Strangeness S*. These are presumbly connected with an invariance property of the Hamiltonian of the strong interactions involving a new co-ordinate not yet understood, and it is further found that the charge of the particle is given by $e\left(M_T + \frac{1}{2}N + \frac{1}{2}S\right)$. We can also consider reactions when they are time dependent ; and here the fundamental result is that if in a reaction process, the interaction Hamiltonian is invariant under some group of transformations, then the initial and final states in the reaction have the same transformation properties with respect to this group. This result can be specialised to particular groups relating to isotopic spin rotations, spatial rotations, and space inversion, and conclusions drawn regarding selection rules, angular distributions and cross-sections.

7. The Lorentz Group & Dirac Equation Relativistic Quantum Theory

The relativistic quantum theory wherein the principles of special relativity and quantum mechanics have been brought into close fusion, has played a fundamental part in the development of quantum theory, specially in the field of elementary particles, and the role of mathematics *via.*, the group theory has been outstanding in this development. This role is played by the Lorentz group of linear co-ordinate transformations connecting two inertial frames x^i and $x^{i'}$ $(i,^{i'} = 1,2,3,4$ with $x^4 = ct)$ of the general form $x^{i'} = a^i + \lambda^{ij} x^j$ leaving the Minkowskian quadratic form $ds^2 = (dx^4)^2 - (dx')^2 - (dx^2)^2 - (dx^3)^2$ invariant. This complete Lorentz group L consists of the important subgroup, called the proper Lorentz group L_4, consisting of rotations in

the four-dimensional $(x^1, x^2, x^3, x^4 = ict)$ space, and excluding the space reflections or inversion P, the time reversal T, and the total inversion which is the product of P and T. The set of elements of P or T do not constitute sub-groups of L, and, moreover, are not of any physical interest by themselves alone since invariance under L_4 is a minimal requirement. Hence, the sub-groups (L_4, PL_4), (L_4, TL_4) and (L_4, PTL_4) each containing L_4 as a proper sub-group of itself, have to be considered to obtain results of physical significance. Irreducible representations of L_4 and of the above three sub-groups can also be obtained and can be used to derive all the finite irreducible representations of the complete Lorentz group L, using Schur's lemma for such representations. It is also necessary to mention here that the notions of the spin of elementary particles, and the celebrated Pauli's rule connecting spin and statistics are two of the most important applications of the theory of the representations of L_4, and they are so well known in the literature that it is hardly worthwhile to derive them here.

The culminating point in the role played by the Lorentz group L is the setting up of a wave equation invariant under it, and this, as is well known, is the famous Dirac equation, $\delta_\mu \beta_\mu \psi + \chi \psi_\mu = 0$ $\left(\delta_u = \dfrac{\partial}{\delta x_\mu}, \mu = 1, 2, 3, 4, \chi = \dfrac{mc}{\hbar}\right)$ which is of fundamental importance in the theory of elementary particles. The invariance of this equation under the complete Lorentz group is sufficient to define the operators β_μ (Dirac matrices) without arbitrariness, and the particles described by this equation have an intrinsic, spin angular momentum, and a magnetic moment. Another remarkable feature of this equation is that which introduces the notion of *charge conjugation* in view of its exhibiting two types of solution which are related by the charge conjugate transformation, or, more precisely, the absence of a particle from (hole in) a negative energy state behaves in every way like the antiparticle, being true even if the particle be a neutral (uncharged) one. We have mentioned above that the β_μ's are determined uniquely by invariance under L, and this implies that the β's satisfy the commutation rule $\beta_\mu \beta_\nu + \beta_\nu \beta_\mu = 2\delta_{\mu\nu}$. Further, this relation can be used to show that the spin of the particle described by the Dirac wave equation is $\frac{1}{2}$ *i.e.*, a fermion.

Thus, in recent years, the Dirac equation is used to represent a single particle of spin $\frac{1}{2}$. In earlier years, some work was done towards using the Dirac equation as representing particles of arbitrary spin which could be characterised by different sets of commutation rules for the β's or, in other words, generalising the algebra generated

by the symbols β_μ and 1 (unit element). In the case of the above Dirac commutation rule, this algebra is the hypercomplex system consisting of the 16 members $1, \beta_\mu, \beta_\mu \beta_\nu, \beta_\lambda \beta_\mu \beta_\nu, \beta_1 \beta_2 \beta_3 \beta_4$ or the Dirac algebra. As is well known, the theorems relating to the representations of a group, can be extended to group rings, and hence to algebras. Applying the theorems which hold in the case of a semi-simple algebra of which the Dirac algebra is a particular case, it is easy to prove that this algebra has only one irreducible representation of order 4, showing the uniqueness of the β_μ's mentioned in the previous paragraph. Altering the Dirac rule to the Duffin commutation rule $\beta_\lambda \beta_\mu \beta_\nu + \beta_\nu \beta_\mu \beta_\lambda = \delta_{\lambda\mu} \beta_\nu + \delta_{\mu\nu} \beta_\lambda$, a theory was built up by Kemmer to show that the associated Dirac equation can represent particles of both the spins 0 and 1, corresponding respectively to the cases of the β's being 5-rowed and 10-rowed square matrices. This arises from the fact that the Duffin-Kemmer algebra has the rank 126, resulting in three representations of orders 1, 5 and 10 satisfying the rank relation $1^2 + 5^2 + 10^2 = 126$. I have myself attempted some years back to set up such algebras for higher spins, in particular spin 3/2, by making the general valid assumptions : (1) the wave equation is invariant under L_4 which leads to the relation $\beta_\mu t_{\mu\nu} - t_{\mu\nu} \beta_\lambda = \delta_{\lambda\mu} \beta_\nu - \delta_{\lambda\nu} \beta_\nu (\mu \neq \nu)$, where $t_{\mu\nu} = i s_{\mu\nu}$ and $s_{\mu\nu}$ is the spin operator. (ii) $t_{\mu\nu} = k(\beta_\mu \beta_\nu - \beta_\nu \beta_\mu) \equiv k(\beta_\mu, \beta_\nu)$, k being a numerical constant, and (iii) each compoent of $s_{\mu\nu}$ satisfies the algebraic equation whose roots are the $(2f+1)$ eigen-values of the spin operator, where f is the spin, and for $f = 3/2$, there are four eigen-values. These assumptions have been used to derive the algebra generated by the symbols, 1 and β_μ, governed by the commutation rules (which can also be obtained from the assumptions made) for the case of spin 3/2 and shown that this algebra is the direct product of the Dirac algebra and another algebra of rank 42, so that the spin 3/2 algebra is of rank 672. Further it is shown that the factor algebra of rank 42 has three irreducible representations of orders, 1, 4 and 5 respectively $(1^2 + 4^2 + 5^2 = 42)$, so that the original algebra has three irreducible representations of orders 4, 16 and 20, the first being identical with the Dirac algebra.

While all this is no doubt elegant mathematics, we might ask the question whether there are really elementary particles of higher spins like 3/2, 2 and so on. In fact, all the 30 elementary particles so far discovered have spins which are either 0, $\frac{1}{2}$ or 1 and so I have recently returned to the question of finding a basis on which these higher spins can be excluded. I have found such a criterion by imposing the condition that one should be able to go over *uniquely*

to a second order wave equation from the first order Dirac type of wave equation. Supposing that the assumption made by Kemmer that the Dirac equation can be used for spins 0 and 1 also *i.e.*, for bosons also (which itself has to be further justified), we can apply the above criterion to Kemmer's case too. By combining the assumption (i) and (ii) of the previous paragraph, we get the general commutation rule $k\ [\beta_\lambda, (\beta_\mu, \beta_\nu)] = \delta_{\lambda\mu}\beta_\nu - \delta_{\lambda\nu}\beta_\mu\ (\mu \neq \nu)$, and the algebraic equations under (iii) are $t^2_{\mu\nu} + \frac{1}{4} = 0$ for spin $\frac{1}{2}$ and $t^3_{\mu\nu} + t_{\mu\nu} = 0$ for spins 0 and 1. The reduction to the second order wave equation $\partial_\mu \partial_\mu \psi = x^2 \psi$ can be next shown to determine k uniquely as equal to $\frac{1}{4}$ or 1 for the cases of spin $\frac{1}{2}$ and spin 0 or 1 respectively. The application of the above method to the two cases of spin 3/2 and 2 does not lead to a unique value of k, but to two alternative values, or in other words, reduction to a fourth order wave equation alone is possible for these higher spins, but a fourth order wave equation has no physical significance. Thus the criterion I have set up definitely excludes the spin 3/2 and higher spins. The result that second order wave equations are uniquely determined for the Kemmer case neither justified taking the Dirac equation as valid for bosons, nor is it necessary to justify the existence of spins 0 and 1 as possible. In fact, if the above argument excluding the spin 3/2 and higher spins be valid, the only alternative possible spins are 0, $\frac{1}{2}$ and 1. I need hardly mention that I do not consider the above criterion for excluding higher spins as being, in any way, rigorous, but it is to be taken as merely suggestive.

8. Relativistic Quantum Theory (contd.): Elementary Particles

We have mentioned in §6 some notions about isotopic spin, strong and weak-interactions, nucleon number and strangeness while dealing with non-relativistic quantum theory. It is easy to show that these notions are valid in relativistic quantum theory also, and we will consider them along with the other notions of parity, charge conjugation and time reversal, also mentioned earlier, in connection with the physics of elementary particles.

Considering charge-conjugation first, it can be generalised to the notion of particle-antiparticle conjugation, since this second type of conjugation does not necessarily imply a change in the sign of the charge, even for a charged particle. Thus the anti-proton $\bar{p}$ which can be defined on Dirac's hole theory mentioned in the previous section, has also a positive electric charge. Also the netural anti-neutron $\tilde{n}$ is distinct from the neutron n in having the opposite

magnetic moment. In the case of bosons, it can be shown that for charged particles, the anti-particle is the one with the opposite charge, while for the neutral boson, the anti-particle may be identical with or different from the original particle. The only two of the former category are the anti-photon $(\bar{\gamma})$ and the anti-neutral pion $\bar{\pi}^0$, since there is no known property by which we could distinguish them from the photon γ, and the neutral pion π^0. On the other hand, the neutral kaon (k – meson) k^0, and its anti particle $\bar{k}^0$ are distinct. Thus we shall mean the more general particle-antiparticle conjugation when we speak of charge conjugation, and denote it by C. The notion of charge conjugation, can be made precise on introducing, besides the Lorentz group, the gauge group of the first kind defined by transformations like $U(x) \rightarrow U(x)\, e^{ia}$ and $U^*(x) \rightarrow U^* e^{-ia}$, where U^* is the complex conjugate of the field quantity U, and α is a constant for free particles. An abbreviation of this theory can also be made so as to apply to neutral particles by identifying charge conjugate states. In the case of particles of zero mass, an additional gauge invariance of another type (called the second kind) enables the building up of a theory for such particles also, *viz.*, the photon γ, and the neutrino, ν^0, the only two particles of zero mass known. A relation can also be set up between the notions of charge-conjugation and isotopic spin in certain cases, as for example, in the case of anti-nucleon annihilation where isotopic spin is conserved, as well as the charge-conjugation eigenvalue.

Coming now to the relations between charge conjugation C, the parity P, and the time reversal T, we might mention the earlier results of Pauli obtained by considering the interactions between particles of spins 0, $\frac{1}{2}$ and 1 *viz.*, (*a*) the weak reflection or the space-time reflection PT holds whether or not the spin-statistics connection is assumed, (*b*) the transformation law for a quantity with respect to L_4 does not determine uniquely its behaviour under PT, and the invariance with respect of this imposes further restrictions on the Lagrangian density of the interactions, besides its invariance under L_4 and (*c*) the strong reflection *i.e.*, the combination of C and PT is uniquely determined as a consequence of L_4 and the spin-statistics connection. The result (*c*) is known as the *Pauli-Luders* theorem. To these results we might add other results obtained by Lee and Yang taking the conservation properties of P and T separately into account *viz.*, that the Pauli-Luders theorem is equivalent to the statements: (*d*) the Hamiltonion H commutes with the product of the operators, P, C and T taken in any order, and (*e*) if H does not commute with P for example, the theory is not invariant under P, and

the same holds for C and T also. Statements (d) and (e) constitute the revolutionary *CPT theorem*, and as a consequence of this theorem it can be shown that if one of the operators P, C, T is not conserved, then at least one other also must not be conserved. Thus there are five possibilities of conservation or non-conservation of P, C, T and these are indicated in the table below :—

No.	Non-conserved operators.	Conserved operators
1	......	P, C, T.
2	C, T.	P, CT., TC
3	P, T.	C, PT., TP
4	C, P.	T, CP. PC
5	P, C, T.	PCT, and permutations.

The CPT theorem of Lee and Yang has recently obtained a brilliant confirmation in experimental results relating to non conservation of parity in the emission of neutrinos in β-decay. Thus in a correct theory of the neutrino ν^0 (defined to be a particle in the positive energy state) its spin is always parallel to its momentum, while the spin of the anti-neutrino ν^0 (defined as a hole in a negative energy state) is always antiparallel to the momentum of ν^0. Thus the spin and velocity of ν^0 represent the spiral motion of a right handed serew, while the spin and velocity of $\bar{\nu}^0$ represent the spiral motion of a left-handed screw. The theory of the neutrino shows that there is no invariance under C, and hence as indicated in the above table, there may be invariance under TCP, and PC (fourth row of the table), or invariance under T also may be violated with invariance under PCT and permutations (5th row of the table). All available evidence up to now, however, appears to show that the position as indicated in the fourth row of the table is the correct one, *i.e.*, that invariance under T is valid.

It would be useful if we were to indicate here the several phenomenological laws of symmetry observed among elementary particles. So far 30 of these have been listed, divided into four groups arranged according to the order of increasing mass. The first group consists only of the photon of zero mass which is a boson of spin 1 identical with its anti-particle. The second group of *leptons*, which are fermions all with spin $\frac{1}{2}$, consists of the neutrino ν^0 of zero mass and the anti-neutrino $\bar{\nu}^0$, the electron e^- and its anti-particle the positron e^+ and

the muon (μ – meson) μ^- and the anti-muon μ^+. The third group of *mesons* which are bosons all with spin 0 consists of the pion π^+, the anti-pion π^- and the neutral pion π^0 identical with its antiparticle $\bar{\pi}^0$, the kaon k^+, the anti kaon k^-, the neutral kaon k^0 and its anti-particle $\bar{k}^0$. Finally, the fourth group of *baroyns* which are fermions all of spin $\frac{1}{2}$ consists of the nucleon p^+ (proton) and its anti-particle $\bar{p}^+$, the nucleon n^0 (neutron) and its anti-particle $\bar{n}^0$, one $\wedge^0$ – hyperon and its anti-particle $\overline{\wedge}_0$, three types of Σ – hyperons, *viz.* Σ_1+ and its anti-particle $\overline{\Sigma}_1+$, Σ_2- and its anti-particle $\overline{\Sigma}_2-$, Σ_3^0 and its anti-particle $\overline{\Sigma}_3^0$, and two types of Ξ – hyperons, *viz.* Ξ_1- and its anti-particie $\overline{\Xi}_1-$, Ξ_2^0 and its anti-particle $\overline{\Xi}_2^0$. The properties that can be measured experimentally are the charge, mass, spin, and life-time if the particle be unstable, and also the decay modes in this case. The particles are assigned three additional quantum numbers *viz.* the baryon, lepton and strangeness quantum numbers B, L & S respectively, assigned for the present on a phenomenological basis only.

Coming now to the phenomenological laws of symmetry, we will mention some of the striking ones based on the presentation due to Marshak (*Science*- 29-7-60, p. 269):

(1) *Spin.*—Regularities among spins are very striking. Except for the photon which is in a class by itself, and possesses spin 1 all the particles in the meson class have spin 0, while all the particles in the lepton and baryon classes posses spin $\frac{1}{2}$.

(2) *Quantum numbers B & L.*—As regards B, the baryons and anti-baryons are assigned $B = \pm 1$ respectively, while all other particles are assigned $B = 0$. As regards I, the leptons and anti-leptons are assigned $L = \pm 1$ respectively, while all other particles are assigned $L = 0$. Conservation laws for B & L and also the charge number $Q(\pm 1$ or $0)$ are absolute conservation laws for all the types, *viz.* strong, electromagnetic and weak interaction. Moreover B, L and Q are additive quantum numbers. Particles which have $B = L = Q = 0$, and also the strangeness $S = 0$ are indistinguishable from their anti-particles, for example the photon, and the neutral poin $\pi^0 \cdot k^0$ and $\bar{k}^0$ also have $B = L = Q = 0$ but, are distinguishable from each other because they have $S = +1$. The conservation of B appears a very subtle phenomenon, and is, in a sense, responsible, more than any other, for the stability of the universe as we know it. Thus for *e.g.* a decay mode like $p \rightarrow 2e^+ + e^-$ is forbidden because of the

conservation of B, and this is justified because this decay mode has a lower limit of 10^{23} years *i.e.*, almost impossible. In a similar manner, absolute conservation of L and Q may also be justified.

(3) *Stable and unstable particles.*—There are only 7 stable particles *viz.* the photon, $e^{+}, e^{-}, \nu^{0}, \bar{\nu}^{0}, p^{+}$ and $\bar{p}^{+}$ while all the rest are unstable taking part in several types of decay modes. Stability of the photon is ensured by the conservations of momentum, energy and angular momentum, that of the nuetrino by the conservation of angular momentum, of the electron by the conservation of Q, and of the proton by the conservation of B. Of the 23 unstable particles, only 3 *viz.*, π^{0}, Σ_{0}, $\overline{\Sigma_{0}}$ emit photons *i.e.* decay *via.* electro magnetic processes. There is at least a factor of 10^{6} in favour of an electromagnetic decay process over a non-electromagnetic decay process over a non-electromagnetic one (*i e.* $10^{-10}: 10^{-16}$), but still 3 out of the 23 take advantage of such a large factor, and this requires an explanation by way of imposing some other kind of a selection rule.

(4) *Strangeness S.*—This quantum number has been introduced to provide the selection rule mentioned above. The observed and unobserved electromagnetic decays among the unstable particles can be understood if we assign a suitable S to each elementary particle, and insist on the conservation of S in all processes involving the electromagnetic force. The assignment is given by $S=0$ for the photon, neutrino, electron, pion, proton and neutron, $S=-1$ for μ^{-} and the Λ and Σ-hyperons, $S=+1$ for k^{+} and k^{0}. and $S=-2$ for the two Ξ-hperons, with the further condition that S for the anti-particle is the negative of S for the particle *i.e.*, like B, L and Q. S is also an additive quantum number. It is remarkable that conservation of S insroduced for electromagnetic interaction should also hold for strong interactions. S is not however, in general conserved in weak interactions. Another consequence of the conservation of S for strong interactions also, is that the production of *strange particles* (*i e.* $S \neq 0$) must take place at least in pairs if the initial state involves only ordinary particles (*i.e.*, $S=0$).

(5) *The three types of forces.*—A dimensionless constant characteristic for each of these types of forces can be set up so as to give a quantitative estimate. For the electro-magnetic type of force, this constant is the fine structure constant $\alpha = e^{2}/4\pi\hbar c = 1/137$. For the strong or nuclear force the analogous constant is $g^{2}/4\pi\hbar c$, where g is the

pion-nucleon coupling constant, which is well determined experimentally, and $g^2/4\pi\hbar c \cong 15$. Unlike as in the case of the strong and electromagnetic forces, where g and e yield the corresponding dimensionless constants, it is not possible to obtain such a constant for the weak force directly. But using the β-decay process, which is the most extensively studied weak interaction, the coupling constant G has the value 1.4×10^{-49}erg $\times$ cm^3, and a dimensionless constant characteristic of weak forces can be obtained from this as $G^2 \times (\hbar c)^{-2} \times (\hbar/\mu c)^{-4} \cong 5 \times 10^{-14}$ where μ is the poin mass, and the order 10^{-14} is not altered by taking some mass other than μ.

(6) *Conservation laws* (*a*)—Conservation laws of energy, momentum, angular momentum, Q, B and L are absolute and valid for all the three types of forces.

(*b*) Conservation of S is valid for strong and $e-m$ forces, but not for weak forces.

(*c*) Conservation of time-reversal T is absolute and holds for all the types of forces.

(*d*) Strong forces are governed by the largest number of conservation laws, with the $e-m$ forces governed by fewer, and the weak forces subjected to the least number. This correlation appears to be one of the important problems in the field of elementary particle physics.

(*e*) The laws of conservation of parity, charge-conjugation and isotopic spin hold at least for one of the forces, but not for all.

(*f*) The strong interactions are invariant under C and also under P, and the same is true of the $e-m$ forces.

(*g*) The *CPT*. theorem, taken in conjunction with (*c*) above indicates conservation of all types of forces under *CP*. This is obvious for strong and $e-m$ forces in view of (*f*), but this behaviour for weak forces which are separately *non-invariant* under C and P is an intriguing problem.

9. Conclusion

The mere enumeration of the phenomenological laws mentioned above is enough to show how complicated the mathematical theories would be, if at all they can be created, to explain these complicated phenomena. We should also expect the possibility of there not existing a single theory, of the type of invariance under L to unite all these theories, for, many of them may be mutually in conflict, and also many of them may be true or false. We might also point out

that, so far, we have not mentioned the theory of gravitation, nor considered the possibility of a quantised theory of gravitation. From a phenomenological point of view this appears unachievable, for the dimensionless constant characterising a type of force discussed in the previous section would give for the gravitational force the constant $kM^2/\hbar c \simeq 2 \times 10^{-39}$, where k is the gravitational constant 6.7×10^{-8} dyne cm^2/gm^2, and M is a large baryon mass. This extraordinarily small number which is of the order 10^{-25} times less than the constant even for weak forces, perhaps justifies the complete neglect of the gravitational force in discussing elementary particles. Even from the mathematical point of view, a union of quantum theory and the theory of gravitation appears difficult of achievement in view of the two theories having roots in mutually exclusive groups of phenomena, and operating with different mathematical concepts—the former with the infinite dimensional Hilbert space, and the latter with the four-dimensional Riemann space. Thus there is no mathematical formalism to which both these theories are approximations. One other characteristic of the quantum theory developed so far is that it has been a purely linear theory. The question arises whether we can set up a non-linear classical field theory, and quantise it, because non-linearity affords many possibilities, while linearity is unique. I have myself dealt with such non-linear classical theories, some years back, based on the Born-Infeld theory, but the problem of quantisation appears impossible of solution. Very recently Dirac has given a reformulation of the Born-Infeld electrodynamics, and refers to the difficulties of quantisation. May it be that the insight of this great physicist suggests such a diversion to the non-linear theories?

It is one of the strangest facts of Nature that she is fond of whole numbers, and in all the quantum phenomena described in the previous articles, integers appear to play a fundamental part in the form of several quantum members governing the phenomena, but the theory of numbers as such, does not appear to have played so far a fundamental role in quantum mechanics.

In conclusion we might point out that the several roles of mathematics, we have indicated above, bring forth clearly the miracle of the appropriateness of the language of mathematics for the formulation of the laws of physics, which we certainly do not understand. But, let us only hope that it will remain valid in future research also, and that mathematics will play still greater roles in unravelling the mysteries of Nature.